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ON (f, g)-DERIVATIONS OF INCLINE ALGEBRAS

Kyung Ho Kim*

Abstract. In this paper, we introduce the concept of a (f, g)-derivation which is a generalization of f-derivation in an incline algebra and give some properties of incline algebras. Also, we consider the *kerd* and k-ideal with respect to (f, g)-derivation in an incline algebra.

1. Introduction

The concept of incline algebra was introduced by Cao and later it was developed by Cao, et.al, in [3]. Recently, a survey on incline algebra was made by Kim and Roush [4, 5]. Incline algebra is a generalization of both Boolean and fuzzy algebra and it is a special type of semiring. It has both a semiring structure and a poset structure. It can also be used to represent automata and other mathematical systems, to study inequalities for non-negative matrices of polynomials. In this paper, we introduce the concept of a (f, g)-derivation which is a generalization of f-derivation in an incline algebra and give some properties of incline algebras. Also, we characterize the *kerd* and k-ideal with respect to (f, g)-derivation in an incline algebra.

2. Incline algebras

An *incline* (*algebra*) is a set K with two binary operations denoted by " $+$ " and " $*$ " satisfying the following axioms: for all $x, y, z \in K$,

(K1) $x + y = y + x$,
(K2) $x + (y + z) = (x + y) + z$,

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(K3) $x * (y * z) = (x * y) * z$,
(K4) $x * (y + z) = (x * y) + (x * z)$,
(K5) $(y + z) * x = (y * x) + (z * x)$,
(K6) $x + x = x$,
(K7) $x + (x * y) = x$,
(K8) $y + (x * y) = y$.

Furthermore, an incline algebra K is said to be *commutative* if $x*y = y*x$ for all $x, y \in K$. Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x * x = x$ for all $x \in K$. Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to see that " $\leq$ " is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an incline algebra K, the following properties hold.

(K9) $x * y \leq x$ and $x * y \leq y$ for all $x, y \in K$,
(K10) $x \leq x + y$ and $y \leq x + y$ for all $x, y \in K$,
(K11) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$, for all $x, y, z \in K$,
(K12) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x * a \leq y * b$ for all $a, b, x, y \in K$.

A *subincline* of an incline algebra K is a non-empty subset M of K which is closed under the addition and multiplication. A subincline M is called an *ideal* if $x \in M$ and $y \leq x$ then $y \in M$. An element "0" in an incline algebra K is a *zero element* if $x+0 = x = 0+x$ and $x*0 = 0 = 0*x$ for any $x \in K$. An non-zero element "1" is called a *multiplicative identity* if $x * 1 = 1 * x = x$ for any $x \in K$. A non-zero element $a \in K$ is called a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such hat $a * b = 0$ (resp. $b * a = 0$) A *zero divisor* is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a homomorphism of inclines, we mean a mapping f from an incline algebra K into an incline algebra L such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. Let K be an incline algebra. An element $a \in K$ is called a *additively left cancellative* if for all $b, c \in K$, $a + b = a + c \Rightarrow b = c$. An element $a \in K$ is called a *additively right cancellative* if for all $b, c \in K$, $b + a = c + a \Rightarrow b = c$. It is said to be *additively cancellative* if it is both left and right cancellative. If every element of K is additively left cancellative, it is called *additively left cancellative.* If every element of K is additively right cancellative, it is called *additively right cancellative.*

3. (f,g)-derivations of incline algebras

Through this article, K stands for an incline algebra with a zero element 0 unless otherwise mentioned.

DEFINITION 3.1. Let K be an incline algebra and let $f, g : K \to K$ be two endomorphisms on K. A self-map d of an incline algebra K is called an (f,g)-*derivation* if it satisfies

$$d(x * y) = d(x) * f(y) + g(x) * d(y) \text{ and } d(x+y) = d(x) + d(y)$$

for every $x, y \in K$.

EXAMPLE 3.2. Let $K = \{0, a, b, 1\}$ be a set in which "+" and " $*$ " is defined by

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, *)$ is an incline algebra. Define a map $d : K \to K$ by

$$d(x) = \begin{cases} a & \text{if } x = a, b, 1 \\ 0 & \text{if } x = 0 \end{cases}$$

and define two endomorphisms $f, g : X \to X$

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b, 1 \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1, a, b \end{cases}$$

Then it is easily checked that d is a (f,g)-derivation of K.

PROPOSITION 3.3. *Let d be a (f,g)-derivation of K. If $f(0) = g(0) = 0$, we have $d(0) = 0$.*

Proof. Let d be a (f,g)-derivation of K. Then

$$\begin{aligned} d(0) &= d(0 * 0) \\ &= (d(0) * f(0)) + (g(0) * d(0)) \\ &= (d(0) * 0) + (0 * d(0)) \\ &= 0 + 0 = 0. \end{aligned}$$

This completes the proof. $\square$

PROPOSITION 3.4. *Let d be a (f, g)-derivation of a commutative incline algebra K. If $f(x) \leq g(x)$ for all $x \in K$, then $d(x) \leq g(x)$ for all $x \in K$.*

Proof. Let $f(x) \leq g(x)$ for all $x \in K$. Then we get $f(x)+g(x) = g(x)$. Thus,

$$\begin{aligned} d(x) &= d(x * x) = (d(x) * f(x)) + (g(x) * d(x)) \\ &= (d(x) * f(x)) + (d(x) * g(x)) \\ &= d(x) * (f(x) + g(x)) \\ &= d(x) * g(x) \leq g(x) \end{aligned}$$

by (K9). This completes the proof. $\square$

PROPOSITION 3.5. *Let d be a (f, g)-derivation of K and let $x, y \in K$ be such that $x \leq y$. Then $d(x * y) \leq g(x) + f(y)$.*

Proof. Let d be a (f, g)-derivation of K and let $x \leq y$. Then we have $d(x) * f(y) \leq f(y)$ and $g(x) * d(y) \leq g(x)$ from (K9). Hence, by using (K12), we get $d(x*y) = (d(x)*f(y))+(g(x)*d(y)) \leq g(x)+f(y)$. This completes the proof. $\square$

PROPOSITION 3.6. *Let d be a (f, g)-derivation of K. Then we have $d(x * y) \leq d(x + y)$ for all $x, y \in K$.*

Proof. Let $x, y \in K$. By using (K9), we get $d(x) * f(y) \leq d(x)$ and $g(x) * d(y) \leq d(y)$. Thus we get

$$d(x * y) = d(x) * f(y) + g(x) * d(y) \leq d(x) + d(y) = d(x + y).$$

$\square$

THEOREM 3.7. *Let f and g be maps on K such that $g(x) \leq f(x)$ for all $x \in K$. If $d = f$ and f, g are two endomorphisms on K, then d is a (f, g)-derivation of K.*

Proof. Let f and g be maps on K such that $g(x) \leq f(x)$ for all $x \in K$. Then we have $g(x) + f(x) = f(x)$ for all $x \in K$. Hence we get

$$\begin{aligned} d(x * y) &= f(x * y) = f(x) * f(y) + f(x) * f(y) \\ &= f(x) * f(y) + (g(x) + f(x)) * f(y) \\ &= f(x) * f(y) + g(x) * f(y) + f(x) * f(y) \\ &= f(x) * f(y) + g(x) * f(y) \\ &= d(x) * f(y) + g(x) * d(y). \end{aligned}$$

Also, we have $d(x + y) = f(x + y) = f(x) + f(y) = d(x) + d(y)$ for all $x, y \in K$. This completes the proof. □

Proposition 3.8. *Let d be a (f, g)-derivation of K. If $d \circ d = d$ and $f \circ d = f$, then $d(x * d(x)) = d(x)$ for all $x \in K$.*

Proof. Let d be a (f, g)-derivation of a distributive lattice K and $d \circ d = d$ and $f \circ d = f$. Then

$$\begin{aligned} d(x * d(x)) &= d(x) * f(d(x)) + g(x) * d(d(x)) \\ &= d(x) * f(x) + g(x) * d(x) \\ &= d(x * x) = d(x) \end{aligned}$$

for all $x \in X$. □

Definition 3.9. Let K be an incline algebra. A mapping f is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

Proposition 3.10. *Let K be an incline algebra and let d be a (f, g)-derivation of K. Then the following identities hold for all $x, y \in K$.*

(i) *$d(x * y) \leq d(x)$ and $d(x * y) \leq d(y)$,*
(ii) *d is isotone.*

Proof. (i) Let $x, y \in K$. Then by using (K7), we obtain

$$d(x) = d(x + (x * y)) = d(x) + d(x * y).$$

Hence we get $d(x*y) \leq d(x)$. Also, $d(y) = d(y+(x*y)) = d(y)+d(x*y)$, and so $d(x * y) \leq d(y)$.

(ii) Let $x \leq y$. Then we have $x + y = y$, and so $d(y) = d(x + y) = d(x) + d(y)$. Hence $d(x) \leq d(y)$. □

Theorem 3.11. *Let M be a nonzero ideal of an integral incline K. If d is a nonzero (f, g)-derivation on K where g is a nonzero function on M, d is a nonzero (f, g)-derivation on M.*

Proof. Assume that g is a nonzero function on M but d is zero (f, g)-derivation on M. Then there is an element $x \in M$ such that $g(x) \neq 0$ and $d(x) = 0$. By (K9), $x * y \leq x$ and since M is an ideal of K, we get $d(x * y) = 0$. Hence we have

$$\begin{aligned} 0 &= d(x * y) = (d(x) * f(y)) + (g(x) * d(y)) \\ &= g(x) * d(y). \end{aligned}$$

Since K has no zero divisors, we have $g(x) = 0$ or $d(y) = 0$. Also, we get $d(y) = 0$ for all $y \in K$ since $g(x) \neq 0$. This contradicts that d is a nonzero (f, g)-derivation on K. Hence d is nonzero on M. □

Let d be a (f, g)-derivation of K. Define a set $Kerd$ by

$$Kerd := \{x \in K \mid d(x) = 0\}.$$

PROPOSITION 3.12. *Let d be a (f, g)-derivation of K. Then $Kerd$ is a subincline of K.*

Proof. Let $x, y \in Kerd$. Then $d(x) = 0, d(y) = 0$ and

$$\begin{aligned} d(x * y) &= d(x) * f(y) + g(x) * d(y) \\ &= 0 * y + x * 0 \\ &= 0 + 0 = 0, \end{aligned}$$

and

$$\begin{aligned} d(x + y) &= d(x) + d(y) \\ &= 0 + 0 = 0. \end{aligned}$$

Therefore, $x * y, x + y \in Kerd$. This completes the proof. □

THEOREM 3.13. *Let d be a (f, g)-derivation of K. Then $Kerd$ is an ideal of K.*

Proof. Let d be a (f, g)-derivation of K. By Proposition 3.12, we know that $Kerd$ is a subincline of K. Let $x \leq y$ and $y \in Kerd$. Then we have $y = x + y$ and $d(y) = 0$. Hence

$$0 = d(y) = d(x + y) = d(x) + d(y) = d(x) + 0 = d(x),$$

which implies $x \in Kerd$, which implies that $Kerd$ is an ideal of K. □

PROPOSITION 3.14. *Let d be a (f, g)-derivation of K. If $x \in Kerd$, we have $x * y \in Kerd$.*

Proof. It is clear from Theorem 3.13 since $x * y \leq x \in Kerd$. This completes the proof. □

Let d be a (f, g)-derivation of K. Define a set $Fix_d(K)$ by

$$Fix_d(K) := \{x \in K \mid f(x) = g(x)\}.$$

PROPOSITION 3.15. *Let d be a (f, g)-derivation of K. Then $Fix_d(K)$ is a subincline of K.*

Proof. Let $x, y \in Fix_d(K)$. Then $f(x) = g(x), f(y) = g(y)$, and so $f(x*y) = f(x)*f(y) = g(x)*g(y) = g(x*y)$ and $f(x+y) = f(x)+f(y) = g(x)+g(y) = g(x+y)$. This implies $x*y, x+y \in Fix_d(K)$. Thus $Fix_d(K)$ is a subincline of K. □

DEFINITION 3.16. A subincline I of an incline algebra K is called a *k-ideal* if $x + y \in I$ and $y \in I$, then $x \in I$.

PROPOSITION 3.17. *Let d be a (f, g)-derivation of an incline algebra K. Then $Kerd$ is a k-ideal of K.*

Proof. In Proposition 3.12, it was showed that $Kerd$ is a subincline of K. Let $x + y \in K$ and $y \in Kerd$. Then we have $d(y) = d(x + y) = 0$. Hence $0 = d(y) = d(x + y) = d(x) + d(y) = d(x) + 0 = d(x)$, which implies that $Kerd$ is a k-ideal of K. $\square$

THEOREM 3.18. *Let d be a (f, g)-derivation of K and let K be additively right cancellative. Then $Fix_d(K)$ is a k-ideal of K.*

Proof. By Proposition 3.15, $Fix_d(K)$ is a subincline of K. Let $x+y \in Fix_d(K)$ and $y \in Fix_d(K)$. Then $g(x) + g(y) = g(x + y) = f(x + y) = f(x) + f(y) = f(x) + g(y)$. Hence we have $f(x) = g(x)$, which implies $x \in Fix_d(K)$. This completes the proof. $\square$

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Department of Mathematics
Korea National University of Transportation
Chungju 380-702, Republic of Korea
E-mail: ghkim@ut.ac.kr